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Inheritance of the Buchsbaum and Cohen–Macaulay Properties in Subcomplexes

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INTRODUCTION

α and γ are topological invariants defined in the work of Munkres (see [7]). Let Σ be a finite abstract simplicial complex, or complex. In the usual way one writes $|\Sigma|$ to denote the realization of Σ. One may fix a field F and define reduced singular cohomology groups $\tilde{H}^i(|\Sigma|, F) = \tilde{H}^i(\Sigma)$ with "F" suppressed for notational simplicity. Note that Σ may be empty in which case $\tilde{H}^i(\phi) = 0$, $i \neq -1$, $\tilde{H}^i(\phi) = F$.

The definition of $\gamma(\Sigma)$ for Σ a complex (see [7, p. 124]) is that $\gamma(\Sigma)$ is the smallest integer j for which at least one of the following groups is not trivial:

$$H^j(|\Sigma|, |\Sigma| - p) \qquad \text{for} \quad p \in |\Sigma|.$$

By convention, $\gamma(\phi) = -1$.

The groups $H^j(|\Sigma|, |\Sigma| - p)$ are called local cohomology groups. In other words, $\gamma(\Sigma)$ is the least integer such that a local cohomology group of dimension $\gamma(\Sigma)$ is non-zero.

The definition of $\alpha(\Sigma)$ may be stated as follows (see [7, p. 116, Theorem 3.1]): $\alpha(\Sigma)$ is the least integer j so that one of the following does not vanish

$$\tilde{H}^j(|\Sigma|), H^j(|\Sigma|, |\Sigma| - p) \qquad \text{for} \quad p \in |\Sigma|.$$

From the definition it is clear that $\gamma(\Sigma) \geqslant \alpha(\Sigma)$ and that $\gamma(\Sigma)$ and $\alpha(\Sigma)$ are topological invariants, i.e., these numbers are not dependent on the particular realization $|\Sigma|$. Let $\dim \Sigma$ denote the dimension of Σ. In general, $\dim \Sigma \geqslant \gamma(\Sigma) \geqslant \alpha(\Sigma) \geqslant -1$. In the case $\alpha(\Sigma) = \dim \Sigma$ ([7, p. 117, Corollary 3.4]), Σ is called Cohen–Macaulay ([1]) with coefficients in F, or CM(F). In the case $\gamma(\Sigma) = \dim \Sigma$, Σ is called Buchsbaum with coefficients in F, or $B(F)$ (see [13]). It should be noted that the first formulations of these concepts occurred in the work of Baclawski who used the term "almost Cohen–Macaulay" instead of "Buchsbaum" ([1, p. 234, Prop. 3.4]). It is clear that if Σ is CM(F) then Σ is $B(F)$. For Σ a complex ([11, Theorem 4.8]) the following can be proven.

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THEOREM 0.1. *If $\alpha = \alpha(\Sigma)$ then the α-skeleton Σ^{α} is the skeleton maximal in the property that Σ^{α} is* CM(F).

Quite similarly, the following can be proved.

THEOREM 0.2. *If $\gamma = \gamma(\Sigma)$ then Σ^{γ} is the skeleton maximal in the property that Σ^{γ} is* B(F).

For Σ a complex, $\alpha(\Sigma)$ and $\gamma(\Sigma)$ are algebraic invariants of a ring called the Stanley–Reisner ring with coefficients in a field F, or $\mathrm{SR}(F, \Sigma)$ (see [9]). Let $V(\Sigma) = \{x_0, ..., x_m\}$ be the vertex set of Σ. Denote by $I(\Sigma)$ the ideal of the polynomial ring $F[X_0, ..., X_m] = S$ generated by all square free monomials of the form $X_{i(1)} \cdots X_{i(k)}$ with the corresponding set $\{x_{i(1)}, ..., x_{i(k)}\} \notin \Sigma$. $\mathrm{SR}(F, \Sigma)$ is defined as $S/I(\Sigma)$. Letting $\mathbf{m} = (X_0, ..., X_m)$, consider the natural homomorphism $\phi : S \to \mathrm{SR}(F, \Sigma)$ and let $M = \phi(\mathbf{m})$.

Let $d = \operatorname{depth}_M \mathrm{SR}(F, \Sigma)$ be the length of the longest regular sequence of $\mathrm{SR}(F, \Sigma)$ within M. It can be proved [11] that $d - 1 = \alpha(\Sigma)$. Thus $d = \operatorname{depth}_M \mathrm{SR}(F, \Sigma)$ is a topological invariant. Using Theorem 0.1 one can state that $d - 1$ is the dimension of the skeleton maximal in the property of being CM(F). This statement generalizes Reisner's theorem [9] when $d = \dim \Sigma + 1$.

Now follows a description of the contents of this paper.

Let $(X, \leqslant)$ be a finite partially ordered set (poset) and regard X as being the order complex where a chain of X corresponds to a simplex of the order complex. If X and Y are posets, then there is a product poset $X \times Y$. It happens that $X \times Y$, the order complex, has realization homeomorphic to the Cartesian product of the realizations of X and Y. Section 1 contains formulas such as $\gamma(X \times Y) = \gamma(X) + \gamma(Y)$ and a corresponding formula (Prop. 1.2) for $\alpha(X \times Y)$.

Section 2 contains the main results of the paper, Theorems 2.1 and 2.2. These statements provide a generalization of a theorem of Munkres ([7, p. 125, (Theorem 6.4]). A partial statement of Theorem 2.1 follows: Let Σ be a complex and $\phi \neq T$ be a subset of vertices of Σ so that if $x \neq x'$ where $x, x' \in T$ then $\{x, x'\} \notin \Sigma$. Let $\Sigma' = \Sigma - \{\sigma \in \Sigma \mid x \in \sigma \text{ for some } x \in T\}$. Then (a) $\alpha(\Sigma') \geqslant \alpha(\Sigma) - 1$.

The proof of (a) in this paper differs from that of [7] in that it uses the Stanley–Reisner ring. The proof is facilitated by a result of Hibi ([4, p. 95, (Theorem]) which allows an exact sequence relating $\mathrm{SR}(F, \Sigma)$ and $\mathrm{SR}(F, \Sigma')$.

Section 3 applies Theorems 2.1 and 2.2 to Cartesian products of posets in case the posets are ranked. In this case the complex of the Cartesian product is pure and a particular decomposition of its realization is possible which behaves predictably with respect to the α and γ invariants.

1. The Behavior of α and γ with Respect to Cartesian Products

Let $X=(X, \leqslant)$ be a finite partially ordered set or poset. X can be regarded as being a complex, the order complex, where a chain $x_1 < x_2 \cdots < x_n$ of length n is regarded as being an $(n-1)$-simplex. For X a complex let $|X|$ denote the realization of X [6].

Let X and Y be posets. Form the Cartesian product poset $X\times Y$ as the set of ordered pairs with the order; for $x, x' \in X$, $y, y' \in Y$, $(x, y) \leqslant (x', y')$ if and only if $x \leqslant x'$ and $y \leqslant y'$. As above, $X\times Y$ can be thought of as the order complex. It is known (for instance, [14]) $|X\times Y| \approx |X| \times |Y|$, the latter the usual Cartesian product.

Proposition 1.1. *Let X and Y be posets. Then $\gamma(X\times Y)=\gamma(X)+\gamma(Y)$.*

Proof. By the relative form of the Eilenberg–Zilber Theorem ([10, p. 234, Theorem 9]) and the Künneth formula ([10, p. 247, Theorem 11]) for $x\in X$, $y\in Y$, and all integers k,

$$H^k(X\times Y, X\times Y-(x, y)) \cong \sum_{i+j=k} H^i(X, X-x) \otimes H^j(Y, Y-y).$$

The result follows immediately. Thanks are given to C. Weibel for the proof.

For a poset X and $\tilde{H}^i(|X|)=\tilde{H}^i(X)$ as above, define the set $A=\{i \mid \tilde{H}^i(X)\neq 0\}$. Define

$$\beta_X=\begin{Bmatrix} \infty, & \text{if } A=\phi \\ \min A, & \text{if } A\neq\phi \end{Bmatrix}.$$

By the definition X is F-acyclic if and only if $\beta_X=\infty$. The next theorem utilizes Proposition 1.1 to compute $\alpha(X\times Y)$.

Proposition 1.2. *Let X and Y be posets. Then $\alpha(X\times Y)=\min\{\gamma(X)+\gamma(Y), \beta_X, \beta_Y\}$.*

Proof. Writing $\beta=\beta_{X\times Y}$, the last theorem gives $\alpha(X\times Y) = \min\{\gamma(X\times Y), \beta\} = \min\{\gamma(X)+\gamma(Y), \beta\}$. But $(*)\ H^k(X\times Y) = \bigoplus_{i+j=k} H^i(X)\otimes_F H^j(Y)$. Clearly, if $\beta_X=\beta_Y=\infty$ then, by $(*)$, $\beta=\infty$ and the result is true. Say $\beta_X=\infty$, $\beta_Y<\infty$, then $\beta=\beta_Y$ whereas $H^{\beta_Y}(X\times Y)$ has a summand $H^{\circ}(X)\otimes_F H^{\beta_Y}(Y)\neq 0$. Similarly, if $\beta_X<\infty$, $\beta_Y=\infty$, then $\beta=\beta_X$. In any case, $\beta=\min\{\beta_X, \beta_Y\}$.

Suppose $\beta_X<\infty$, $\beta_Y<\infty$. By $(*)$, $H^{\beta_X}(X\times Y)$ has a non-trivial summand $H^{\beta_X}\otimes_F H^{\circ}(Y)$ and $H^{\beta_Y}(X\times Y)$ has a non-trivial summand $H^{\circ}(X)\otimes_F H^{\beta_Y}(Y)$. From definitions of β_X and β_Y it is clear that $\beta=\min\{\beta_X, \beta_Y\}$ and the proof is complete.

The next two consequences improve on statements of Baclawski [1, p. 249, Theorem 7.1].

COROLLARY 1.3. *Let X and Y be posets. The following are equivalent*:

(a) *X and Y are* $\mathrm{B}(F)$,

(b) *$X \times Y$ is* $\mathrm{B}(F)$.

Proof. Suppose X and Y are Buchsbaum with coefficients in the field F. Then $\gamma(X) = \dim X$, $\gamma(Y) = \dim Y$. By Proposition 1.1, $\gamma(X \times Y) = \gamma(X) + \gamma(Y) = \dim X + \dim Y$. Whereas $\dim(X \times Y) = \dim X + \dim Y$, the argument for (a) implies (b) is finished. The converse is similar.

The next corollary has a homotopy Cohen–Macaulay formulation by Walker (see [14, Theorem 7.1]).

COROLLARY 1.4. *Let X and Y be posets. The following are equivalent.*

(a) *$X \times Y$ is* $\mathrm{CM}(F)$.

(b) *X and Y are* $\mathrm{B}(F)$ *and either X and Y are both acyclic or both are of dimension* 0.

Proof. Suppose (a). For $X \times Y$ to be $\mathrm{CM}(F)$, X and Y are $\mathrm{B}(F)$ by Corollary 1.3. Also, $\alpha(X \times Y) = \dim(X \times Y) = \dim X + \dim Y$. By Proposition 1.2,

$$\tilde{H}^i(X) = 0 = \tilde{H}^i(Y) \qquad \text{for} \quad i < \dim X + \dim Y. \tag{$*$}$$

Assuming that X and Y are not both zero dimensional, $\dim X + \dim Y > 0$. Equation (*) now states that X and Y are acyclic and (b) is proved.

Now suppose (b). Then $\gamma(X) = \dim X$, $\gamma(Y) = \dim Y$, and by Proposition 1.2, $\alpha(X \times Y) = \min\{\dim X + \dim Y, \beta_X, \beta_Y\}$. In each case of (b), $\alpha(X \times Y) = \dim X + \dim Y$ and $X \times Y$ is $\mathrm{CM}(F)$.

2

In this section fix a field F and write for a complex Σ, $\mathrm{SR}(F, \Sigma) = \mathrm{SR}(\Sigma)$. Let $V(\Sigma)$ be the vertex set of Σ.

Suppose $\Sigma' \subseteq \Sigma$ is a subcomplex. There is a natural homogeneous ring homomorphism of zero degree $\varphi\colon \mathrm{SR}(\Sigma) \to \mathrm{SR}(\Sigma')$ induced by defining

$$\varphi(X_1 X_2 \cdots X_n + I(\Sigma)) = \left\{ \begin{matrix} 0, & \text{if } \{x_1, ..., x_n\} \notin V(\Sigma') \\ X_1 X_2 \cdots X_n + I(\Sigma'), & \text{otherwise} \end{matrix} \right\},$$

$$\text{for} \quad \{x_1, ..., x_2\} \in V(\Sigma).$$

Let $R = \mathrm{SR}(\Sigma)$. $\mathrm{SR}(\Sigma')$ can be made into an R-module by projection : $r \cdot s = \varphi(r)s$ for $r \in R$ and $s \in \mathrm{SR}(\Sigma')$. Then φ can be regarded as an R-module homomorphism.

If M is the homogeneous maximal ideal of $R = \mathrm{SR}(\Sigma)$, M is the ideal of R generated by elements of the form $X_1 \cdots X_n + I(\Sigma)$, and M' is the homogeneous maximal ideal of $\mathrm{SR}(\Sigma')$, then $\varphi(M) = M'$. As a result, if $\mathrm{SR}(\Sigma')$ is regarded as an R-module, then $\mathrm{depth}_M \mathrm{SR}(\Sigma') = \mathrm{depth}_{M'} \mathrm{SR}(\Sigma')$, where the latter depth is the length of the longest regular sequence of M' within $\mathrm{SR}(\Sigma')$ the ring [5]. This last equality is used without comment in what follows.

Record from [11] for later use:

Note 2.1. Let Σ be a simplicial complex and M be the homogeneous maximal ideal of $\mathrm{SR}(\Sigma)$. Then $\mathrm{depth}_M \mathrm{SR}(\Sigma) - 1 = \alpha(\Sigma)$.

Define for Σ a simplicial complex and $\sigma \in \Sigma$ the following subcomplexes:

$$\mathrm{link}(\sigma, \Sigma) = \{\tau \in \Sigma \mid \sigma \cap \tau = \phi \text{ and } \sigma \cup \tau \in \Sigma\},$$
$$\mathrm{star}(\sigma, \Sigma) = \{\tau \in \Sigma \mid \sigma \cup \tau \in \Sigma\}.$$

LEMMA 2.2. *Let Σ be a complex and x be a vertex of Σ. Then $\alpha(\mathrm{star}(x, \Sigma)) \geqslant \gamma(\Sigma) \geqslant \alpha(\Sigma)$.*

Proof. Whereas $\gamma(\Sigma) \geqslant \alpha(\Sigma)$ is true by definition, it suffices to prove $\alpha(\mathrm{star}(x, \Sigma)) \geqslant \gamma(\Sigma)$. By [7, Corollary 6.3], $\alpha(\mathrm{link}(x, \Sigma)) \geqslant \min_{x \in V(\Sigma)} \alpha(\mathrm{link}(x, \Sigma)) = \gamma(\Sigma) - 1$, so $\alpha(\mathrm{link}(X, \Sigma)) + 1 \geqslant \gamma(\Sigma)$. The lemma follows from the fact that $\alpha(\mathrm{link}(x, \Sigma)) + 1 = \alpha(\mathrm{star}(x, \Sigma))$. The proof of the last equality is deferred till Section 3 (see Corollary 3.2).

The following theorem generalizes [7, p. 125, Theorem 6.4] in that no requirement is made that every maximal $\sigma \in \Sigma$ must contain some $x \in T$.

THEOREM 2.3. *Let Σ be a complex and $T \subseteq V(\Sigma)$ be a subset such that, for $x, x' \in T$ with $x \neq x'$, $\{x, x'\} \notin \Sigma$. Let Σ' be the subcomplex $\Sigma' = \Sigma - \{\sigma \in \Sigma \mid \sigma$ contains some $x \in T\}$. Then*

(a) $\alpha(\Sigma') \geqslant \alpha(\Sigma) - 1$.

(b) *If $\alpha(\Sigma') = \alpha(\Sigma) - 1$ or $\alpha(\Sigma') > \alpha(\Sigma)$, then $\alpha(\mathrm{star}(x, \Sigma)) = \alpha(\Sigma)$ for some $x \in T$ and $\gamma(\Sigma) = \alpha(\Sigma)$.*

(c) *If $\alpha(\mathrm{star}(x, \Sigma)) > \alpha(\Sigma)$ for each $x \in T$, then $\alpha(\Sigma') = \alpha(\Sigma)$.*

Proof. Let $R = \mathrm{SR}(\Sigma)$ and consider the exact sequence of R-modules of [4, p. 95]:

$$0 \to \bigoplus_{x \in T} \mathrm{SR}(\mathrm{star}(x, \Sigma)) \to R \to \mathrm{SR}(\Sigma') \to 0. \tag{1}$$

If M is the homogeneous maximal ideal of R and $\operatorname{Ext}^i_R(R/M, -)$ is the usual derived functor, then one has the long exact sequence

$$\cdots \to \operatorname{Ext}^i_R(R/M, L) \to \operatorname{Ext}^i_R(R/M, R) \to \operatorname{Ext}^i_R(R/M, \mathrm{SR}(\Sigma')), \qquad (2)$$

where $L = \bigoplus_{x \in T} \mathrm{SR}(\mathrm{star}(x, \Sigma))$.

Equation (2) allows for the measurement of depth by the fact of [5, Theorem 28];

$$\operatorname{depth}_M H = n \Leftrightarrow \operatorname{Ext}^i_R(R/M, H) \begin{cases} = 0 & \text{for } i < n \\ \neq 0 & \text{for } i = n \end{cases}$$

for $H = L, R$, or $\mathrm{SR}(\Sigma')$. Because $\operatorname{Ext}^i_R(R/M, L) \simeq \bigoplus_{x \in T} \operatorname{Ext}^i(R/M, \mathrm{SR}(\mathrm{star}(x, \Sigma)))$, it follows that

$$\operatorname{depth}_M L = n \Leftrightarrow \operatorname{depth}_M \mathrm{SR}(\mathrm{star}(x, \Sigma)) \geqslant n \qquad \text{for each} \quad x \in T \qquad (3)$$

and

$$\operatorname{depth}_M \mathrm{SR}(\mathrm{star}(x, \Sigma)) = n \qquad \text{for some} \quad x \in T.$$

For (a) use Lemma 2.2 to conclude that $\alpha(\mathrm{star}(x, \Sigma)) \geqslant \alpha(\Sigma)$ for each $x \in T$. By Note 2.1 and (3),

$$\operatorname{depth}_M L \geqslant \alpha(\Sigma) + 1. \qquad (4)$$

Conclude from (4) and (2) that $\operatorname{Ext}^i_R(R/M, \mathrm{SR}(\Sigma')) = 0$ for all $i \leqslant \alpha(\Sigma) - 1$. So $\operatorname{depth}_M \mathrm{SR}(\Sigma') \geqslant \alpha(\Sigma)$ and, by Note 2.1, $\alpha(\Sigma') \geqslant \alpha(\Sigma) - 1$.

For (b) suppose $\alpha(\Sigma') = \alpha(\Sigma) - 1$. By Note 2.1 and (2), $\operatorname{Ext}^i_R(R/M, L) = 0$ for all $i \leqslant \alpha$ and there is an exact sequence

$$0 \to \operatorname{Ext}^\alpha_R(R/M, \mathrm{SR}(\Sigma')) \neq 0 \to \operatorname{Ext}^{\alpha+1}_R(R/M, L).$$

It follows that $\operatorname{depth}_M L = \alpha$ and there is by (3) some $x \in T$ such that $\operatorname{depth}_M \mathrm{SR}(\mathrm{star}(x, \Sigma)) = \alpha + 1$. By Note 2.1, $\alpha(\mathrm{star}(x, \Sigma)) = \alpha$ and, by Lemma 2.2, (b) follows in part. For the other part where $\alpha(\Sigma') > \alpha(\Sigma)$ the argument is similar and is omitted.

Now assume the hypothesis of (c). Then by (3), $\operatorname{depth}_M L > \alpha(\Sigma) + 1$. From (2) it follows that $\operatorname{Ext}^i_R(R/M, \mathrm{SR}(\Sigma')) \cong \operatorname{Ext}^i_R(R/M, \mathrm{SR}(\Sigma))$ for all $i \leqslant \alpha(\Sigma) + 1$. By Note 2.1, $\alpha(\Sigma') = \alpha(\Sigma)$. This completes the proof.

Now let Σ and Σ' be as above under the assumption that $\gamma(\Sigma) > \alpha(\Sigma)$. Lemma 2.2 allows the statement that $\alpha(\mathrm{star}(x, \Sigma)) > \alpha(\Sigma)$ for each $x \in T$. By (c) in Theorem 2.3, $\alpha(\Sigma') = \alpha(\Sigma)$. This argument proves the following corollary which generalizes a statement in [7] (Corollary 6.5, p. 126).

COROLLARY 2.4. *Let Σ and σ' be as above. If $\gamma(\Sigma) > \alpha(\Sigma)$ then* $\alpha(\Sigma') = \alpha(\Sigma)$.

Now consider a counterpart statement for $\gamma(\Sigma)$ and $\gamma(\Sigma')$.

THEOREM 2.5. *Let Σ be a complex and a subset such that for $x, x' \in T$ with $x \neq x'$, then $\{x, x'\} \notin \Sigma$. Let Σ' be the subcomplex $\Sigma' = \Sigma - \{\sigma \in \Sigma \mid \sigma$ contains some $X \in T\}$. Then*

(a) $\gamma(\Sigma') \geqslant \gamma(\Sigma) - 1$.

(b) *If* $\gamma(\Sigma') = \gamma(\Sigma) - 1$ *then there is* $y \in V(\Sigma')$ *such that* $\alpha(\mathrm{link}(y, \Sigma')) = \gamma(\Sigma) - 2$ *and* $\alpha(\mathrm{link}(y, \Sigma)) = \gamma(\Sigma) - 1$.

(c) *If* $\alpha\ (\mathrm{star}(x, \Sigma)) > \gamma(\Sigma)$ *for each* $x \in T$ *then* $\gamma(\Sigma') = \gamma(\Sigma)$.

Proof. It is known that

$$\min_{y \in V(\Sigma')} \alpha(\mathrm{link})(y, \Sigma')) = \gamma(\Sigma') - 1 \tag{$+$}$$

(see [7]). For (a) it suffices to prove that

$$\alpha(\mathrm{link}(y, \Sigma')) \geqslant \gamma(\Sigma) - 2 \qquad \text{for each} \quad y \in V(\Sigma'). \tag{$\bigcirc$}$$

Now let $y \in V(\Sigma')$ and consider by [11] the following short exact sequence of R-modules, $R = \mathrm{SR}(\Sigma)$:

$$\begin{aligned} 0 \to \bigoplus_{\substack{x \in T \\ \{x\} \in \mathrm{link}(y, \Sigma)}} & \mathrm{SR}(\mathrm{star}(x, \mathrm{link}(y, \Sigma))) \to \mathrm{SR}(\mathrm{link}(y, \Sigma)) \\ & \to \mathrm{SR}(\mathrm{link}(y, \Sigma')) \to 0. \end{aligned} \tag{1}$$

By Lemma 2.2,

$$\alpha(\mathrm{star}(x, \mathrm{link}(y, \Sigma))) \geqslant \alpha(\mathrm{link}(y, \Sigma)) \qquad \text{for} \quad x \text{ as in (1)}. \tag{2}$$

Consider the long exact sequence of R-modules

$$\begin{aligned} \cdots \to \mathrm{Ext}^i_R(R/M, L) \to & \mathrm{Ext}^i_R(R/M, \mathrm{SR}(\mathrm{link}(y, \Sigma))) \\ & \to \mathrm{Ext}^i_R(R/M, \mathrm{SR}(\mathrm{link}(y, \Sigma'))) \to \cdots, \end{aligned} \tag{3}$$

where

$$L = \bigoplus_{\substack{x \in T \\ \{x\} \in \mathrm{link}(y, \Sigma)}} \mathrm{SR}(\mathrm{star}(x, \mathrm{link}(y, \Sigma))).$$

As in the proof of Theorem 2.3,

$$\mathrm{depth}_M L = n \Leftrightarrow \mathrm{depth}_M \mathrm{SR}(\mathrm{star}(x, \mathrm{link}(y, \Sigma'))) \geqslant n \qquad \text{for all} \quad x \in T \tag{4}$$

and $\operatorname{depth}_M \mathrm{SR}(\operatorname{star}(x, \operatorname{link}(y, \Sigma'))) = n$ for some $x \in T$. By (2), (4), Note 2.1, and (+),

$$\operatorname{depth}_M L \geqslant \operatorname{depth}_M \mathrm{SR}(\operatorname{link}(y, \Sigma)) = \alpha(\operatorname{link}(y, \Sigma)) + 1 \geqslant \gamma(\Sigma). \tag{5}$$

From (3) follows $\operatorname{Ext}^i_R(R/M, \mathrm{SR}(\operatorname{link}(y, \Sigma'))) = 0$ for all $i < \gamma(\Sigma) - 2$; i.e., $\operatorname{depth}_M \mathrm{SR}(\operatorname{link}(y, \Sigma')) \geqslant \gamma(\Sigma) - 1$. Eq. (○) follows by Note 2.1.

In order to prove (b) assume $\gamma(\Sigma') = \gamma(\Sigma) - 1$. By (+) choose $y \in V(\Sigma')$ such that $\gamma(\Sigma') = 1 + \alpha(\operatorname{link}(y, \Sigma'))$. Then $\gamma(\Sigma) - 2 = \alpha(\operatorname{link}(y, \Sigma'))$. Use this y in the construction of sequences (1) and (3). By Note 2.1 there exists an exact sequence:

$$0 \to \operatorname{Ext}^{\gamma(\Sigma)-1}_R(R/M, \mathrm{SR}(\operatorname{link}(y, \Sigma'))) \neq 0 \to \operatorname{Ext}^{\gamma(\Sigma)}_R(R/M, L). \tag{6}$$

As in (4), for some $x \in T$, $\operatorname{Ext}^{\gamma(\Sigma)}_R(R/M, \mathrm{SR}(\operatorname{star}(x, \operatorname{link}(y, \Sigma)))) \neq 0$. By (3), $\operatorname{Ext}^i_R(R/M, \mathrm{SR}(\operatorname{star}(x, \operatorname{link}(y, \Sigma)))) = 0$ for each $i < \gamma(\Sigma)$. So $\operatorname{depth}_M \mathrm{SR}(\operatorname{star}(x, \operatorname{link}(y, \Sigma))) = \gamma(\Sigma))$ and $\alpha(\operatorname{star}(x, \operatorname{link}(y, \Sigma)) = \gamma(\Sigma) - 1$. By (+) and Lemma 2.2, $\alpha(\operatorname{link}(y, \Sigma)) \geqslant \min_{x \in V(\Sigma)} \alpha(\operatorname{link}(x, \Sigma)) = \gamma(\Sigma) - 1 = \alpha(\operatorname{star}(x, \operatorname{link}(y, \Sigma))) \geqslant \alpha(\operatorname{link}(y, \Sigma))$. Part (b) is now established.

For (c) assume $\alpha(\operatorname{star}(x, \Sigma)) > \gamma(\Sigma)$ for each $x \in T$. As in the proof of Lemma 2.2, $\alpha(\operatorname{star}(x, \Sigma)) = \alpha(\operatorname{link}(x, \Sigma)) + 1$, so the assumption is

$$\alpha(\operatorname{link}(x, \Sigma)) > \gamma(\Sigma) - 1 \qquad \text{for each} \quad x \in T. \tag{7}$$

As in (+), (7) yields

$$\gamma(\Sigma) - 1 = \min_{y \in V(\Sigma')} \alpha(\operatorname{link}(y, \Sigma)). \tag{8}$$

Pick $y \in V(\Sigma')$. Let Case 1 be the situation where there does not exist an $x \in T$ with $\{x\} \in \operatorname{link}(y, \Sigma)$. In Case 1, $\operatorname{link}(y, \Sigma) = \operatorname{link}(y, \Sigma')$. In Case 2 there is $x \in T$ with $\{x\} \in \operatorname{link}(y, \Sigma)$. For this x, $\operatorname{link}(x, \operatorname{link}(y, \Sigma)) = \operatorname{link}(\{x, y\}, \Sigma) = \operatorname{link}(y, \operatorname{link}(x, \Sigma))$. By (7) and Lemma 2.2,

$$\alpha(\operatorname{star}(x, \operatorname{link}(y, \Sigma))) = \alpha(\operatorname{star}(y, \operatorname{link}(x, \Sigma))) \geqslant \alpha(\operatorname{link}(x, \Sigma)) > \gamma(\Sigma) - 1. \tag{9}$$

Now from sequences (1) and (3) for y of Case 2. From (9), as earlier in the proof,

$$\operatorname{depth}_M L > \gamma(\Sigma). \tag{10}$$

Because of (8),

$$\operatorname{depth}_M \mathrm{SR}(\operatorname{link}(y, \Sigma)) \geqslant \gamma(\Sigma). \tag{11}$$

Now using (3), (10), and (11) implies

$$\operatorname{depth}_M \operatorname{SR}(\operatorname{link}(y, \Sigma')) \geqslant \gamma(\Sigma), \tag{12}$$

and this inequality is true for all $y \in V(\Sigma')$, including y of Case 1.

Choose $y_0 \in V(\Sigma')$ with $\alpha(\operatorname{link}(y_0, \Sigma)) = \gamma(\Sigma) - 1$ by (8). By (3), Note 2.1, and (10), $\operatorname{depth}_M \operatorname{SR}(\operatorname{link}(y_0, \Sigma')) = \gamma(\Sigma)$. Using (12), it is clear that

$$\min_{y \in V(\Sigma')} \operatorname{depth}_M \operatorname{SR}(\operatorname{link}(y, \Sigma') = \gamma(\Sigma).$$

By Note 2.1 and (+),

$$\gamma(\Sigma') - 1 = \min_{y \in V(\Sigma')} \alpha(\operatorname{link}(y, \Sigma')) = \gamma(\Sigma) - 1.$$

Conclude $\gamma(\Sigma') = \gamma(\Sigma)$.

3. Applications

The primary goal of this section is to apply Theorems 2.3 and 2.4 to the direct product poset $X \times Y$.

For X and Y posets define the joint poset $X * Y$ to be the disjoint union of X and Y with order restricting to the given orderings on X and Y so that any element of X is less than any element of Y. Considering $X * Y$ as order complex, then $|X * Y| \approx |X| * |Y|$ where $|X| * |Y|$ represents the union of line segments joining a point of $|X|$ to a point of $|Y|$ where it is required that any two of the line segments intersect at most at an endpoint. It is useful later that for X, Y, Z posets $|X * Y| \approx |Y * X|$ and $|X| * |Y * Z| \approx |X * Y| * |Z|$ (see [6, p. 371, Lemma 62.4]).

Let $X = \{x_0\}$ and Y be posets. Then $|X * Y|$ is the usual cone over $|Y|$. Let $X = \{x_0, x_1\}$ and Y be posets with x_0 and x_1 unrelated elements. Then $|X * Y|$ is the usual suspension of $|Y|$. It is convenient to write $|X * Y| = |S^0| * |Y| = \operatorname{susp} |Y|$.

The proposition below can be established by means of direct calculation using simplicial cohomology (see [2, p. 165]).

Proposition 3.1. *Let X and Y be posets. Then $\alpha(X * Y) = \alpha(X) + \alpha(Y) + 1$.*

Let X be a poset. Considering X as the order complex, form $By(X)$ where $By(X)$ is the poset of all non-empty simplices of X with order the reverse of inclusion. It is well known that $|By(X)| \approx |X|$ whereas $|By(X)|$ is the barycentric subdivision of $|X|$. This notation is used in the proof of the following corollary which is in turn needed for the proof of Lemma 2.2.

COROLLARY 3.2. *Let X be a complex and x be a vertex of X. Then* $\alpha(\text{star}(x, X)) = \alpha(\text{link}(x, X)) + 1$.

Proof. $|\text{star}(x, X)| \approx |By(\text{star}(x, X))|$ so it suffices to compute $\alpha(By(\text{star}(x, X)))$. There is a straightforward poset isomorphism $By(\text{star}(x, X)) \cong By(\text{link}(x, X)) * \{x\}$. By Proposition 3.1, $\alpha(By(\text{star}(x, X))) = \alpha(By(\text{link}(x, X))) + 1 = \alpha(\text{link}(x, X)) + 1$.

Let τ be a finite set. Denote by $C(\tau)$ the set of all subsets of τ (including ϕ). Then $C(\tau)$ is the complex of the simplex τ. If $C(\tau)$ is made into the poset $\underline{X}(C(\tau))$ using reverse inclusion order, the usual Boolean lattice occurs. Let $\sigma \subseteq \tau$; i.e., $\sigma \in C(\tau)$. Then $(\tau, \sigma) = \{w \in C(\tau) \mid \tau < w < \sigma\}$ is a subposet of $\underline{X}(C(\tau))$.

In the sequel for $N \geqslant 0$ the notation S^N denotes a complex which has a topological N-sphere as realization. By convention, $S^{-1} = \phi$.

LEMMA 3.3. *Let τ be a set and $\sigma \in C(\tau)$. With notation as above,* $(\tau, \sigma) = S^{\dim \tau - \dim \sigma - 2}$.

Proof. By assumption $\sigma \subseteq \tau$. Let $r = \dim \tau - \dim \sigma$. There is an isomorphism of posets

$$[\tau, \sigma] \cong \underline{X}(C(\tau - \sigma)) = \underline{X}.$$

This is easy to see as one may subtract σ from each element of the poset $[\tau, \sigma]$ to arrive at $\underline{X}$.

But $(\tau, \sigma) \cong \underline{X} - \{\tau - \sigma, \phi\}$ and the poset $\underline{X} - \{\tau - \sigma, \phi\}$ is the poset of nontrivial simplices of the boundary complex of an $(r-1)$-simplex $\tau - \sigma$. So the complex $(\tau, \sigma) = S^{r-2} = S^{\dim \tau - \dim \sigma - 2}$.

Given a poset X and $x_0 \times X$ write $X_{>x_0} = \{x \in X \mid x > x_0\}$, $X_{<x_0} = \{x \in X \mid x < x_0\}$.

The way is now prepared for the following technical statement where $|X \times Y| \approx |X| \times |Y| \approx |By(X)| \times |By(Y)| \approx |By(X) \times By(Y)|$.

PROPOSITION 3.4. *For X and Y posets and $\Sigma = \Sigma(By(X) \times By(Y))$ the order complex, let $\phi \neq \tilde{w} \in \Sigma$, $\tilde{w} = \{(\sigma_0, \tau_0) < \cdots < (\sigma_k, \tau_k)\}$. Then*

$$|\text{link}(\tilde{w}, \Sigma)| \approx |\text{link}(\sigma_0, X)| * |\text{link}(\tau_0, Y)| * |S^{\dim \sigma_0 + \dim \tau_0 - k - 1}|,$$

X and Y regarded as order complexes.

Proof. Note that

$$\begin{aligned} \text{link}(\tilde{w}, \Sigma) = {} & By(X) \times By(Y)_{<(\sigma_0, \tau_0)} * ((\sigma_0, \tau_0), (\sigma_1, \tau_1)) \\ & * \cdots \quad \cdots * ((\sigma_{k-1}, \tau_{k-1}), (\sigma_k, \tau_k)) * By(X) \\ & \times By(Y)_{>(\sigma_k, \tau_k)}. \end{aligned} \tag{0}$$

Consider the pieces of the join poset. First $|By(X) \times By(Y)_{<(\sigma_0, \tau_0)}| \approx |By(X)_{<\sigma_0}| * |By(Y)_{<\sigma_0}|$ by a result of Quillen ([8, Prop. 1.9]). The poset $By(X)_{<\sigma_0}$ is isomorphic to the poset consisting of non-empty simplices of link (σ_0, X) with order reverse-inclusion. It follows that $By(X)_{<\sigma_0}$ is the poset whose realization is the barycentric subdivision of $\mathrm{link}(\sigma_0, X)$, so $|By(X)_{<\sigma_0}| \approx |\mathrm{link}(\sigma_0, X)|$. Likewise $|By(Y)_{<\tau_0}| \approx |\mathrm{link}(\tau_0, Y)|$. It follows that

$$|By(X) \times By(Y)_{<(\sigma_0 \tau_0)}| \approx |\mathrm{link}(\sigma_0, X)| * |\mathrm{link}(\tau_0, Y)|. \tag{1}$$

Again by [8], $|By(X) \times By(Y)_{>(\sigma_k, \tau_k)}| \approx |By(X)_{>\sigma_k}| * |By(Y)_{>\tau_k}|$. But $By(X)_{>\sigma_k}$ is the poset of proper non-empty subsets of σ_k ordered by reverse inclusion. Thus $|By(X)_{>\sigma_k}| = |S^{\dim \sigma_k - 1}|$. Also, $|By(Y)_{>\tau_k}| = |S^{\dim \tau_k - 1}|$. But

$$|S^{\dim \sigma_k - 1}| * |S^{\dim \tau_k - 1}| \approx |S^{\dim \sigma_k - 1} * S^{\dim \tau_k - 1}| \approx |S^{\dim \sigma_k + \dim \tau_k - 1}|,$$

the latter homeomorphism obtained by an easy induction. So follows

$$|By(X) \times By(Y)_{>(\sigma_k, \tau_k)}| \approx |S^{\dim \sigma_k + \dim \tau_k - 1}|. \tag{2}$$

Now fix some i, $0 \leqslant i \leqslant k-1$. By a result of [14, Theorem 5.1] and Lemma 3.3,

$$\begin{aligned}
|((\sigma_i, \tau_i), ((\sigma_{i+1}, \tau_{i+1}))| &\approx \mathrm{susp}(|(\sigma_i, \sigma_{i+1})| * |(\tau_i, \tau_{i+1})|) \\
&\approx |S^0| * |S^{\dim \sigma_i - \dim \sigma_{i+1} - 2}| * |S^{\dim \tau_i - \dim \tau_{i+1} - 2}| \\
&\approx |S^0| * |S^{\dim \sigma_i - \dim \sigma_{i+1} - 2} * S^{\dim \tau_i - \dim \tau_{i+1} - 2}| \\
&\approx |S^0| * |S^{\dim \sigma_i - \dim \sigma_{i+1} + \dim \tau_i - \dim \tau_{i+1} - 3}| \\
&\approx |S^{\dim \sigma_i - \dim \sigma_{i+1} + \dim \tau_i - \dim \tau_{i+1} - 2}|.
\end{aligned}$$

It follows from (0), (1), and (2) that

$$\begin{aligned}
|\mathrm{link}(\bar{w}, \Sigma)| &\approx |\mathrm{link}(\sigma_0, X)| * |\mathrm{link}(\tau_0, Y)| * |S^{\dim \sigma_0 - \dim \sigma_0 - \dim \tau_1 - 2}| \\
&\quad * \cdots * |S^{\dim \sigma_{k-1} - \dim \sigma_k + \dim \sigma_{k-1} - \dim \tau_k - 2}| * |S^{\dim \sigma_k + \dim \tau_k - 1}| \\
&\approx |\mathrm{link}(\sigma_0, X)| * |\mathrm{link}(\tau_0, Y)| * |S^{\dim \sigma_0 + \dim \tau_0 - \dim \sigma_k - \dim \tau_k - k - 1}| \\
&\quad * |S^{\dim \sigma_k + \dim \tau_k - 1}| \\
&\approx |\mathrm{link}(\sigma_0, X)| * |\mathrm{link}(\tau_0, Y)| * |S^{\dim \sigma_0 + \dim \tau_0 - k - 1}|
\end{aligned}$$

and the proof is complete.

Let X be a poset. X is ranked of rank n if the length of every maximal chain within X is n. Note that if X is ranked of rank n, then the order complex X is pure of dimension $n-1$, i.e., every maximal simplex of X has dimension $n-1$.

Let X and Y be ranked posets with $\dim X = M$ and $\dim Y = N$. Then $Z = By(X) \times By(Y)$ is a pure order complex of dimension $M+N$. For i, with $0 \leqslant i \leqslant M+N$, let $Z_i = Z - \{(\sigma, \tau) \in Z) \mid \dim \sigma + \dim \tau > M+N-i\}$. Note that Z_i is a subcomplex of Z of dimension $M+N-i$, and $Z_0 = Z$. There is a decomposition $Z = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_{M+N} \supseteq Z_{M+N+1} = \phi$. The following two theorems show invariants α and γ to behave regularly with respect to this decomposition.

THEOREM 3.5. *Let X and Y be ranked posets with* $\dim X = M$ *and* $\dim Y = N$ *and* $Z = By(X) \times By(Y)$. Let $\alpha(X \times Y) = \alpha(Z) = M+N-k$ *for some k with $0 \leqslant k \leqslant M+N$. Then with notation as above,*

(a) *for i with $0 \leqslant i \leqslant k$, $\alpha(Z_i) = \alpha(Z)$.*

(b) *For i with $i \geqslant k$, $\alpha(Z_i) = M+N-i$. So for $i \geqslant k$, $\alpha(Z_i)$ is* CM(F).

Proof. For (a) one may assume $\alpha(Z) = M+N-k$ with $k > 0$. The proof is by induction on i.

For base step $i=1$ let $T = \{(\sigma, \tau) \in Z \mid \dim \sigma + \dim \tau = M+N\}$. T consists of the set of atoms of Z. Let $(\sigma, \tau) \in T$. By Proposition 3.4, $|\mathrm{link}((\sigma, \tau), Z)| \approx |S^{M+N-1}|$, so $\alpha(\mathrm{link}((\sigma, \tau), Z)) = M+N-1$. By Corollary 3.2, $\alpha(\mathrm{star}((\sigma, \tau), Z)) = M+N$. As $(\sigma, \tau) \in T$ is arbitrary, the fact that no two elements of T are comparable in Z proves via Theorem 2.3(c) that $\alpha(Z_1) = \alpha(Z)$.

For the inductive step, assume $0 < i \leqslant k$ and $\alpha(Z_{i-1}) = M+N-k = \alpha(Z)$. Consider $Z_i \subseteq Z_{i-1}$ and let $T_i = \{(\sigma, \tau) \in Z \mid \dim \sigma + \dim \tau = M+N-i+1\}$. τ_i is the set of atoms of Z_{i-1}. Let $(\sigma, \tau) \in T_i$. By purity of Z there exists a chain $(\sigma_0, \tau_0) < (\sigma_1, \tau_1) < \cdots < (\sigma_{i-1}, \tau_{i-1}) = (\sigma, \tau)$ where for each l, $\dim \sigma_l + \dim \tau_l = M+N-l$. Let $\bar{w} = \{(\sigma_0, \tau_0), (\sigma_1, \tau_1), \ldots, (\sigma_{i-1} \tau_{i-1}\} \in Z$. It is clear that $\mathrm{link}(\bar{w}, Z) = \mathrm{link}((\sigma, \tau), Z_{i-1})$. Now $|\mathrm{link}(\bar{w}, Z)| \approx |S^{M+N-(i-1)-1}| = |S^{M+N-i}|$ by Proposition 3.4. As in the base step, $\alpha(\mathrm{link}((\sigma, \tau), Z_{i-1})) = M+N-i+1 > M+N-k = \alpha(Z_{i-1})$. The elements of T_i are not comparable in Z_{i-1}. It follows by Theorem 2.3(c) that $\alpha(Z_i) = \alpha(Z_{i-1}) = \alpha(Z)$, and thus (a) is proven by induction.

For (b) it suffices to prove the first statement. So assume $\alpha(Z_k) = M+N-k$ and let $T_{k+1} = \{(\sigma, \tau) \in Z \mid \dim \sigma + \dim \tau = M+N-k\}$. T_{k+1} is the set of atoms of Z_k. Then $Z_{k+1} = Z_k - \{\bar{w} \in Z_{k+1} \mid (\sigma, \tau) \in \bar{w}$ for some $(\sigma, \tau) \in T_{k+1}\}$. As in the above proof of (a), Theorem 2.3(a) yields $\alpha(Z_{k+1}) \geqslant \alpha(Z_k) - 1 = M+N-k-1$. But $\dim Z_{k+1} = M+N-k-1$. Conclude $\alpha(Z_{k+1}) = M+N-k-1$. (b) is now established by an easy induction.

Note that (b) above can be proved by using [1, Theorem 6.4, p. 247].

THEOREM 3.6. *Let X and Y be ranked posets with* $\dim X = M$ *and* $\dim Y = N$ *and* $Z = By(X) \times By(Y)$. *Let* $\gamma(Z) = M + N - k$ *for some* k *with* $0 \leqslant k \leqslant M + N$. *Then*

(a) *for* i *with* $0 \leqslant i \leqslant k$, $\gamma(Z_i) = \gamma(Z)$.

(b) *For* i *with* $i \geqslant k$, $\gamma(Z_i) = M + N - i$.

So for $i \geqslant k$, $\gamma(Z_i)$ *is* $\mathrm{B}(F)$.

Proof. The proof is almost identical with the proof of Theorem 3.5 except an appeal is made to Theorem 2.5 instead of Theorem 2.3.

It should be pointed out that the decomposition $Z = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_{M+N+1} = \phi$ is essentially different from the decomposition of skelatons $Z \supseteq Z^{M+N-1} \supseteq \cdots \supseteq Z^0 \supseteq \phi$. In the simple case $|X| = |Y| = I^1$, the unit intervals Z_1 and Z^1 are not even of the same homotopy type. But is is known that results similar to Theorems 3.5 and 3.6 can be stated in terms of the second decomposition. More generally, one has

PROPOSITION 3.6 ([11, Theorem 4.8, Ptop. 3.6]). *Let Z be a poset with* $\dim Z = N$ *and* $\alpha(Z) = N - k$ *for* k *with* $0 \leqslant k \leqslant N(\gamma(Z) = N - k)$. *Then*

(a) *for* i *with* $0 \leqslant i \leqslant k$, $\alpha(Z^i) = \alpha(Z)(\gamma(Z^i) = \gamma(Z))$.

(b) *For* i *with* $i \geqslant k$, $\alpha(Z^i) = N - i(\gamma(Z^i) = N - i)$.

Proof. Part (a)'s proof is based on Theorems 0.1 and 0.2 of the Introduction. Part (b) can be proven by using Theorems 2.3 and 2.5 or [4] (see p. 96) or [1] (see p. 247).

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